

# Multiplications of Distributions in One Dimension and a First Application to Quantum Field Theory

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In a previous paper we introduced a class of multiplications of distributions in one dimension. Here we furnish different generalizations of the original definition and we discuss some applications of these procedures to the multiplication of delta functions and to quantum field theory. © 2002 Elsevier Science (USA)

## 1. INTRODUCTION

In the past years many attempts have been made to extend the ordinary multiplication between functions to distributions. Of course, as for the extensions of (almost) any kind, the procedure is not unique and, in fact, many inequivalent proposals are present nowadays in the literature; see [1–3], among others.

In this paper we generalize a class of multiplication of distributions introduced in previous work by the author; see [4]. The original definition was based on two different regularizations of distributions, the analytic and the sequential completion methods. In particular, this last procedure makes reference to functions in  $\mathcal{D}(\mathbf{R})$  which generate the so-called delta families. A delta family is essentially a set of functions which approximate  $\delta(x)$  in the topology of  $\mathcal{D}(\mathbf{R})$ . Sometimes, whenever the applications require it, it may appear necessary to use a weaker form of this procedure. Possible weakening of the requirements in [4] are part of the content of this paper. In particular, in Section 2 we relax some of the requirements given in [4], so that, in principle, more distributions can be multiplied between themselves,

while in Section 3 we give two inequivalent multiplications among more than two distributions.

The necessity for extending the multiplication to more than two distributions follows directly from physical examples: this is what we need to do whenever we try to regularize three- or four-point Green's functions in a given quantum field model. In fact, it has been recognized since Wightmann's work [5] that the field operators are not operator-valued functions, but rather distributions defined on a certain domain dense in a Hilbert space. We know also that a field theory is often defined via a lagrangian density,  $\mathcal{L}$ , which depends on the products of such fields considered at coincident points. This is, of course, an operation which has no rigorous mathematical meaning in this naive form. One of the most famous consequences of this procedure is that certain Feynman diagrams diverge, and that analogous divergences are observed also in many matrix elements of the dynamical variables. Many attempts have been made in the past decades to give a rigorous meaning to quantum field theory (QFT). This has generated essentially two different approaches: constructive QFT, as proposed first by Wightmann, where the Wightmann functions, that is some "matrix elements" of the fields, are the relevant dynamical variables, and the algebraic QFT, developed by Haag and Kastler in the 1960s, which we will not consider here. In [5] a sort of regularization procedure for the Wightmann functions is widely discussed: they can be recovered as the boundary values of some holomorphic functions. Nevertheless, even using this regularization, the problem of the divergences in a perturbative QFT still exists and the solutions proposed are, in our opinion, unsatisfactory.

A possible way out is to compute these Feynman graphs using the regularizations of the fields, instead of the fields themselves, and to remove the regularization only at the end. This is the path we will follow in Section 5. The results we will obtain, however, show once more the difficulty of the problem: in particular we will see that our regularization procedure, as it is, is not powerful enough to avoid the appearance of the divergences in a free QFT in  $1 + 1$  dimensions.

An idea close to ours is also behind Colombeau's book [6], where a systematic approach to QFT is proposed. The lack of uniqueness in the regularization procedure makes Colombeau's work not resolute. For instance, other approaches, more along the lines of this paper, can be found in [2, 7].

The paper is divided as follows: In the next section we start recalling the definition of the multiplication given in [4]. We take also the opportunity for briefly discussing some new result. Then we "relax" this definition of the multiplication to better deal with physical models involving distributions of  $\mathcal{S}'$ , like in QFT.

In Section 3 we discuss two possible generalizations of the theory discussed in Section 2 to more than two distributions. In Section 4 we show

many examples involving delta functions of both the generalizations proposed. We also discuss some physical examples. In Section 5 we apply our procedure to an easy quantum field model, the free  $1 + 1$  Klein–Gordon theory. In the last section, finally, we comment the results and state a plan for the future.

## 2. DEFINITION OF THE MULTIPLICATION

In this section we briefly recall, for readers' convenience, the basic definitions and results of the multiplication introduced in [4]. We will slightly modify this definition later, so as to build up a framework which is more suitable for applications to QFT.

Let  $\mathcal{V}$  be the subspace of all the functions in  $C^\infty$  with arbitrary support,  $\mathcal{E}$ , with the following properties:

- (i)  $\phi(x) |x| \leq k_0$  for  $|x| \rightarrow \infty$ ,
- (ii)  $\phi^{(n)}(x) |x| \leq k_n$  for  $x \rightarrow \infty$ ,

where  $k_0, k_1, \dots$  are constants. The convergence is defined as in  $\mathcal{E}$ .

Let  $\mathcal{V}'$  be the dual space of  $\mathcal{V}$ . For distributions in this space it has been shown in [2] that the function

$$\mathbf{T}^0(z) \equiv \frac{1}{2\pi i} T \cdot (x - z)^{-1} \quad (2.1)$$

exists and is holomorphic in  $z$  in the whole  $z$ -plane minus the support of  $T$ . The function

$$T_{red}(x, \epsilon) \equiv \mathbf{T}^0(x + i\epsilon) - \mathbf{T}^0(x - i\epsilon) \quad (2.2)$$

is further weakly convergent to the distribution  $T$  when  $\epsilon$  goes to zero [2]. Also, if  $T(x)$  is a continuous function with compact support, then  $T_{red}(x, \epsilon)$  converges uniformly to  $T(x)$  on the whole real axis for  $\epsilon \rightarrow 0^+$ .

The other ingredient of the multiplication in [4] is the method of the sequential completion, which makes reference to the so-called  $\delta$ -sequences. In [4] we have called a  $\delta$ -sequence a sequence of functions  $\delta_n(x) \equiv n\phi(nx)$ , where  $\phi \in \mathcal{D}(\mathbf{R})$  is a given function with  $\text{supp } \phi \subseteq [-1, 1]$  and  $\int \phi(x)dx = 1$ . Then,  $\forall T \in \mathcal{D}'(\mathbf{R})$ , the convolution  $T_n \equiv T * \delta_n$  is a  $C^\infty$ -function, for any fixed  $n \in \mathbf{N}$ . The sequence  $T_n$  converges to  $T$  in the topology of  $\mathcal{D}'$ , when  $n \rightarrow \infty$ . Moreover, if  $T(x)$  is a continuous function with compact support then  $T_n(x)$  converges uniformly to  $T(x)$ .

In [4] we have proceeded in the following way: for any couple of distributions  $T, S \in \mathcal{V}'$ ,  $\forall \alpha, \beta > 0$  and  $\forall \Psi \in \mathcal{D}$  we have defined the quantity

$$(S \otimes T)_n^{(\alpha, \beta)}(\Psi) \equiv \frac{1}{2} \int_{-\infty}^{\infty} \left[ S_n^{(\beta)}(x) T_{red} \left( x, \frac{1}{n^\alpha} \right) + T_n^{(\beta)}(x) S_{red} \left( x, \frac{1}{n^\alpha} \right) \right] \Psi(x) dx, \quad (2.3)$$

where  $S_n^{(\beta)}(x) \equiv (S * \delta_n^{(\beta)})(x)$ , with  $\delta_n^{(\beta)}(x) \equiv n^\beta \Phi(n^\beta x)$ . Hence, the two distributions  $S$  and  $T$  in  $\mathcal{V}'$  are said to be multipliable if the limit of  $(S \otimes T)_n^{(\alpha, \beta)}$  for  $n \rightarrow \infty$  exists finite in a weak sense. Finally, we have defined

$$(S \otimes T)_{(\alpha, \beta)}(\Psi) \equiv \lim_{n \rightarrow \infty} (S \otimes T)_n^{(\alpha, \beta)}(\Psi). \quad (2.4)$$

In [4] we have proved, among other things, that this product extends the usual product of the functions, in the sense that if  $T(x)$  and  $S(x)$  are two continuous functions with compact supports then the product  $T_n^{(\beta)}(x) S_{red}(x, 1/n^\alpha)$  converges uniformly to  $T(x) S(x)$ . As mathematical applications of our definition we have discussed the possibility of multiplying two (derivatives of) delta functions localized at the same point. Here we want to make this information complete. In particular we want to extend the multiplication to arbitrary derivatives of  $\delta(x)$ ,  $\delta^{(k)}$ . Since this result is a straightforward generalization of what has been done in [4], we will not give all the details. If we want to define  $(\delta^{(k)} \otimes \delta^{(l)})_{(\alpha, \beta)}$  then we are forced to consider different situations depending on the parity of the integers  $k$  and  $l$ . We get

$$(\delta^{(k)} \otimes \delta^{(l)})_{(\alpha, \beta)} = \begin{cases} 0 & \alpha > (k+l+2)\beta \\ \frac{(k+l+1)!}{\pi} A_{k+l+2} \delta, & \alpha = (k+l+2)\beta \quad k, l \text{ even} \\ -\frac{(k+l+1)!}{\pi} A_{k+l+2} \delta, & \alpha = (k+l+2)\beta \quad k, l \text{ odd} \\ 0 & \alpha = (k+l+2)\beta \quad k \text{ even and } l \text{ odd,} \end{cases} \quad (2.5)$$

where we have taken, as in [4],

$$\Phi(x) = \begin{cases} \frac{x^m}{F} \cdot \exp \left\{ \frac{1}{x^2 - 1} \right\}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases} \quad (2.6)$$

Here  $F$  is a normalization constant, and we have defined, whenever they exist,  $A_j \equiv \int_{-\infty}^{\infty} (\Phi(x)/x^j) dx$ , for integers  $j$ . In order to have a finite regularization (2.5), we always have to choose a function  $\Phi(x)$  with  $m$  even and

such that  $m > k + l + 1$ . It is interesting to notice that Eq. (2.5) implies, among others, the equalities

$$\begin{aligned}(\delta'' \otimes \delta'')_{(\alpha, \beta)} &= -(\delta' \otimes \delta''')_{(\alpha, \beta)} = (\delta \otimes \delta'')_{(\alpha, \beta)} \\ (\delta' \otimes \delta')_{(\alpha, \beta)} &= -(\delta \otimes \delta'')_{(\alpha, \beta)},\end{aligned}$$

which show that, at least for this particular example, the usual properties of the derivatives of the distributions are satisfied by the product  $\otimes_{(\alpha, \beta)}$ .

As we have already discussed in the Introduction, we are interested in applying our proposal of regularization and multiplication to quantum fields, which are operators whose matrix elements belong to  $\mathcal{S}'$ . It is therefore natural to generalize a bit the above definition, trying to construct a framework closer to the physics. In particular, we modify the definition of the sequential completion, which is strongly related, in its original version, to distributions in  $\mathcal{D}'$ . There exist also technical reasons which suggest relaxing the definition of the sequential completion. We will comment on this point in Section 5.

Using the terminology of [8], we call delta sequence of Dirichelet type a family of functions  $\delta_k(x)$ , which satisfy the following conditions:

- (i)  $\int_{-A}^A \delta_k(x) dx \rightarrow 1$  when  $k \rightarrow \infty$  for a certain  $A > 0$ ;
- (ii)  $\forall \gamma > 0, \forall f \in \mathcal{L}^1(\mathbf{R})$  then  $\lim_{k \rightarrow \infty} (\int_{-\infty}^{-\gamma} + \int_{\gamma}^{\infty}) (\delta_k(x) f(x) dx) = 0$ ;
- (iii)  $\exists C_1, C_2$ , positive constants:  $|\delta_k(t)| \leq C_1/|t| + C_2$ .

In particular, it is an easy exercise to prove that for any  $\Phi(t)$  belonging to  $\mathcal{S}$ , such that  $\int_{-\infty}^{\infty} \Phi(t) dt = 1$ , then the family of functions  $\delta_n^{(\beta)}(x) \equiv n^\beta \Phi(n^\beta x)$  generates a delta sequence of Dirichelet type if  $\beta > 0$ .

We have the following:

**PROPOSITION 1.** *Let  $T \in \mathcal{S}'(\mathbf{R})$  and let  $\delta_n^{(\beta)}(x)$  be a delta sequence of Dirichelet functions. Then the convolution  $T_n^{(\beta)} \equiv T * \delta_n^{(\beta)}$  is a  $C^\infty$ -function, for any fixed  $n \in \mathbf{N}$ . The sequences  $\delta_n^{(\beta)}(x)$  and  $T_n^{(\beta)}(x)$  converge respectively to  $\delta$  and to  $T$  in the topology of  $\mathcal{S}'$ , when  $n \rightarrow \infty$ , for all  $\beta > 0$ .*

*Moreover, if  $T(x)$  is a Hölder-continuous function with compact support  $[a, b]$ , then  $T_n^{(\beta)}(x)$  converges uniformly to  $T(x)$  on every interior subinterval of  $[a, b]$ .*

*Proof.* The  $C^\infty$ -nature of the function  $T_n^{(\beta)}(x)$  is a well known property of the convolutions, which can be found, for instance, in [9].

In [8] it is proved that any  $\delta$ -sequence of Dirichelet type weakly converges to the  $\delta$  function with respect to any function which has a finite derivative in the origin. A fortiori therefore  $\delta_n^{(\beta)}$  will converge to  $\delta$  in the topology of  $\mathcal{S}'$ . From this fact the convergence of  $T_n^{(\beta)}$  to  $T$  in  $\mathcal{S}'$  easily follows.

The last statement is again contained in [8]. ■

*Remarks.* (a) Another generalization of the  $\delta$ -family is also discussed in [8]. The family is now called a *delta family of positive type* and, as the name itself suggests, its functions must all be not negative. This is, in general, a strong requirement which is not necessarily satisfied by the “generating” function  $\Phi(t)$  we will use in the application to QFT, and this is the reason we have focused our attention to Dirichelet type functions. Nevertheless, even for such a delta family a proposition like the one above can be stated; minor differences are required in the hypotheses but the results, essentially, coincide.

(b) Any  $\Phi(t) \in \mathcal{D}(\mathbf{R})$  generating a “standard” delta family also generates a delta family of Dirichelet type and, if  $\Phi(x) \geq 0$  for all  $x \in \mathbf{R}$ , also a delta family of positive type.

(c) One may wonder why we have introduced so many families of delta functions: the reason is that the choice of the function  $\Phi$  cannot be made in general a priori by us, but it is often forced by the model which has to be regularized. In particular, in the example of QFT we will show that there is no reason, in general, for  $\Phi(x)$  to have compact support or to be positive.

(d) Proposition 1 can be used to show that the new multiplication still extends the usual multiplication of continuous functions with compact support, in the sense that if  $T(x)$  and  $S(x)$  are Hölder-continuous functions with compact support in  $[a, b]$  then,  $\forall \alpha, \beta > 0$  and  $\forall \Psi \in \mathcal{S}$ ,

$$(T \otimes S)_{(\alpha, \beta)}(\Psi) = \int_{-\infty}^{\infty} T(x)S(x)\Psi(x) dx.$$

The definition of the multiplication is now, formally, the same as Eqs. (2.3) and (2.4). The only difference is in the mathematical nature of  $\delta_n^{(\beta)}$ .

We end this section by giving the extension of the definition of multiplication to the case in which the distributions  $S$  and  $T$  do not commute, even if we will not meet with this problem in this paper. In this condition we are forced to symmetrize the original definition (2.3), (2.4). Let  $S$  and  $T$  be two operator valued distributions. Keeping the same notation as before, we define

$$\begin{aligned} & (S \otimes T)_{(\alpha, \beta)}(\Psi) \\ & \equiv \frac{1}{4} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ S_n^{(\beta)}(x) T_{red} \left( x, \frac{1}{n^\alpha} \right) + T_n^{(\beta)}(x) S_{red} \left( x, \frac{1}{n^\alpha} \right) \right. \\ & \quad \left. + T_{red} \left( x, \frac{1}{n^\alpha} \right) S_n^{(\beta)}(x) \right. \\ & \quad \left. + S_{red} \left( x, \frac{1}{n^\alpha} \right) T_n^{(\beta)}(x) \right] \Psi(x) dx. \end{aligned} \quad (2.7)$$

Of course this definition must be understood in the weak (Hilbert) sense. Moreover, whenever  $S$  and  $T$  commute (again in the weak (Hilbert) sense), the above definition returns the original one.

### 3. MULTIPLYING MORE DISTRIBUTIONS

Up to now we have focused our interest on the multiplication of two distributions and their possible definitions. This is not enough in many physical situations, for instance, in the computation of the four-point Green's functions in a scalar  $\lambda\varphi^4$  theory, [10]. In this perspective we will now analyze possible extensions of the definition (2.4) when more than two distributions are considered. In particular we will suggest two different, inequivalent, approaches, and we will discuss some examples. Which method has to be chosen only depends on which one gives theoretical results in (a better) agreement with the experiments (or with common sense). We will return to this point with an example at the end of the next section.

In this paper we will consider only commuting distributions. This is a useful condition to simplify all formulas.

The first method we are going to discuss is, in our opinion, the most natural one since it does not need any new ingredients to be defined. We start with two distributions  $S_1$  and  $S_2$ . Their multiplication, if it exists, is defined by (2.4). Let us now suppose to be interested in defining the product of three distributions  $S_1, S_2$ , and  $S_3$  in  $\mathcal{V}'$ . It is quite natural to consider the quantity

$$\begin{aligned} (S_1 \otimes S_2 \otimes S_3)_n^{(\alpha, \beta)} \\ \equiv \frac{1}{3} \left[ (S_1 \otimes S_2)_n^{(\alpha, \beta)} S_3 + (S_1 \otimes S_3)_n^{(\alpha, \beta)} S_2 + (S_2 \otimes S_3)_n^{(\alpha, \beta)} S_1 \right], \end{aligned} \quad (3.1)$$

which is certainly well defined for any fixed  $n$ , since any term above is the product of a  $C^\infty$  function for a distribution. As usual, what may or may not exist is the limit for  $n \rightarrow \infty$  of  $(S_1 \otimes S_2 \otimes S_3)_n^{(\alpha, \beta)}(\Psi)$ , for any  $\Psi \in \mathcal{D}(\mathbf{R})$ . If this limit exists we say that the distributions can be multiplied and we put

$$(S_1 \otimes S_2 \otimes S_3)_{(\alpha, \beta)}(\Psi) \equiv \lim_{n \rightarrow \infty} (S_1 \otimes S_2 \otimes S_3)_n^{(\alpha, \beta)}(\Psi). \quad (3.2)$$

It is useful to notice that formula (3.1) would look rather more complicated without the working hypothesis of the commutativity of the distributions.

Let us now try to define a multiplication between four distributions. In this case, of course, we cannot repeat the same steps leading to Eq. (3.2), since the quantity  $(S_1 \otimes S_2 \otimes S_3)_{(\alpha, \beta)} S_4$  would necessarily contain the product of two un-regularized distributions. We have to define this multiplication

in a different way. This problem can be easily overcome simply by coupling the distributions in all the possible ways and then using twice the regularization. This implies that the product of four distributions should depend on four indices, two  $\alpha$ 's and two  $\beta$ 's, Explicitly we have

$$\begin{aligned} & (S_1 \otimes S_2 \otimes S_3 \otimes S_4)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)}(\Psi) \\ & \equiv \lim_{n \rightarrow \infty} \frac{1}{6} \left\{ (S_1 \otimes S_2)_n^{(\alpha_1, \beta_1)} (S_3 \otimes S_4)_n^{(\alpha_2, \beta_2)} + (S_1 \otimes S_3)_n^{(\alpha_1, \beta_1)} \right. \\ & \quad \times (S_2 \otimes S_4)_n^{(\alpha_2, \beta_2)} + (S_1 \otimes S_4)_n^{(\alpha_1, \beta_1)} (S_2 \otimes S_3)_n^{(\alpha_2, \beta_2)} \\ & \quad \left. + (\alpha_1, \beta_1) \leftrightarrow (\alpha_2, \beta_2) \right\}(\Psi), \end{aligned} \quad (3.3)$$

whenever this limit exists. To be more explicit, for instance the first term of this formula reads

$$\begin{aligned} & \left[ (S_1 \otimes S_2)_n^{(\alpha_1, \beta_1)} (S_3 \otimes S_4)_n^{(\alpha_2, \beta_2)} \right](\Psi) \\ & = \int_{-\infty}^{\infty} S_{1,n}^{(\beta_1)}(x) S_{2,red} \left( x, \frac{1}{n^{\alpha_1}} \right) S_{3,n}^{(\beta_2)}(x) S_{4,red} \left( x, \frac{1}{n^{\alpha_2}} \right) \Psi(x) dx. \end{aligned} \quad (3.4)$$

The multiplication of five distributions is now naturally defined in analogy with the one in (3.1) and (3.2). We put

$$\begin{aligned} & (S_1 \otimes S_2 \otimes S_3 \otimes S_4 \otimes S_5)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)}(\Psi) \\ & \equiv \frac{1}{5} \left[ (S_1 \otimes S_2 \otimes S_3 \otimes S_4)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} S_5 + (S_1 \otimes S_2 \otimes S_3 \otimes S_5)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} \right. \\ & \quad \times S_4 + (S_1 \otimes S_2 \otimes S_4 \otimes S_5)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} S_3 \\ & \quad + (S_1 \otimes S_3 \otimes S_4 \otimes S_5)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} S_2 \\ & \quad \left. + (S_2 \otimes S_3 \otimes S_4 \otimes S_5)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} S_1 \right](\Psi), \end{aligned}$$

whenever the right hand side exists for any  $\Psi(x) \in \mathcal{D}(\mathbf{R})$ .

It is clear now how this procedure can be generalized to the multiplication of an arbitrary number  $N$  of distributions: whenever  $N$  is even we have to proceed like in (3.3); that is, we consider all the different  $N/2$  pairs of distributions, regularize each pair, and then try to remove the regularization. If  $N$  is odd, we simply have to multiply one un-regularized distribution with the regularization of the even  $N - 1$  remaining ones.

In all the examples discussed in this work we will stick to the situation in which all the distributions coincide. In this case all the formulas are strongly simplified. Whenever the limits below exist we have

$$(S \otimes S)_{(\alpha, \beta)} = \lim_{n \rightarrow \infty} S_n^{(\beta)}(x) S_{red} \left( x, \frac{1}{n^\alpha} \right) \quad (3.5)$$

$$(S \otimes S \otimes S)_{(\alpha, \beta)} = S \lim_{n \rightarrow \infty} (S \otimes S)_n^{(\alpha, \beta)} \quad (3.6)$$



$$(S \otimes S \otimes S \otimes S)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} = \lim_{n \rightarrow \infty} (S \otimes S)_n^{(\alpha_1, \beta_1)} (S \otimes S)_n^{(\alpha_2, \beta_2)} \quad (3.7)$$

$$(S \otimes S \otimes S \otimes S \otimes S)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} = S \lim_{n \rightarrow \infty} (S \otimes S)_n^{(\alpha_1, \beta_1)} (S \otimes S)_n^{(\alpha_2, \beta_2)}, \quad (3.8)$$

and so on. The generalization to a bigger number of distributions is straightforward. All the formulas above are obviously thought of their weak forms: they must be *applied* to a generic function  $\Psi \in \mathcal{D}$ , like in Eq. (3.4).

We now discuss a different proposal which again extends the multiplication introduced in (2.4) for two distributions.

First of all, let us introduce two complex quantities  $a_1, a_2$ , with  $a_1 + a_2 = 2$ . We modify the original definition (2.4) of the multiplication of two distributions by saying that *two distributions  $S_1$  and  $S_2$  are  $A$ -multipliable if there exists a choice of  $a_1$  and  $a_2$ , with  $a_1 + a_2 = 2$ , such that the following limit exists*:

$$\begin{aligned} (S_1 \otimes S_2)_{(\alpha, \beta)}^A(\Psi) \equiv & \frac{1}{2} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ a_1 S_{1,n}^{(\beta)}(x) S_{2,red} \left( x, \frac{1}{n^\alpha} \right) \right. \\ & \left. + a_2 S_{1,red} \left( x, \frac{1}{n^\alpha} \right) S_{2,n}^{(\beta)}(x) \right] \Psi(x) dx, \end{aligned} \quad (3.9)$$

for any  $\Psi \in \mathcal{D}(\mathbf{R})$ .

Of course definition (2.4) turns out to be simply a special case of this one when we take  $a_1 = a_2 = 1$ . It is interesting to notice that our new multiplication depends now not only on  $\alpha, \beta$ , but also on  $a_1$  and  $a_2$ . Of course, it may happen that one contribution in (3.9) does not converge for  $n \rightarrow \infty$ . In this case, while the multiplication in (2.4) is not defined, the one above still exists for a clever choice of  $a_1$  and  $a_2$ .

The length of the formulas rapidly increases when the number of distributions to be multiplied grows. Already for three distributions we need to introduce six parameters,  $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}$ , and  $a_{23}$ , whose sum must be equal to 6. The  $A$ -multiplication of the three distributions is said to exist if there exists a choice of the coefficients  $a_{ij}$ 's and of the pair  $(\alpha, \beta)$  such that the limit below exists,

$$\begin{aligned} (S_1 \otimes S_2 \otimes S_3)_{(\alpha, \beta)}^A(\Psi) \\ \equiv & \frac{1}{6} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ a_{11} S_{1,n}^{(\beta)}(x) S_{2,n}^{(\beta)}(x) S_{3,red} \left( x, \frac{1}{n^\alpha} \right) + a_{12} S_{1,n}^{(\beta)}(x) S_{2,red} \left( x, \frac{1}{n^\alpha} \right) \right. \\ & \times S_{3,n}^{(\beta)}(x) + a_{13} S_{1,red} \left( x, \frac{1}{n^\alpha} \right) S_{2,n}^{(\beta)}(x) S_{3,n}^{(\beta)}(x) \\ & \left. + a_{21} S_{1,n}^{(\beta)}(x) S_{2,red} \left( x, \frac{1}{n^\alpha} \right) S_{3,red} \left( x, \frac{1}{n^\alpha} \right) \right. \end{aligned}$$

$$\begin{aligned}
& + a_{22} S_{1, red} \left( x, \frac{1}{n^\alpha} \right) S_{2, n}^\beta(x) S_{3, red} \left( x, \frac{1}{n^\alpha} \right) \\
& + a_{23} S_{1, red} \left( x, \frac{1}{n^\alpha} \right) S_{2, red} \left( x, \frac{1}{n^\alpha} \right) S_{3, n}^{(\beta)}(x) \Big] \Psi(x) dx, \quad (3.10)
\end{aligned}$$

for all  $\Psi \in \mathcal{D}$ .

In the case of four distributions the number of the coefficients grows to 14, so that it is more and more difficult to correctly take into account all these contributions. However, the situation drastically simplifies when all the distributions coincide. In this case we have symmetry arguments which give some extra conditions on the coefficients  $a$ .

For instance, in the case of two equal distributions, from definition (3.9) it is evident that we have to take  $a_1 = a_2 = 1$ . Hence, this method returns the usual result; see Eq. (3.5).

From (3.10) we deduce that, if  $S_1 = S_2 = S_3 = S$ , then necessarily  $a_{11} = a_{12} = a_{13} =: b_1$  and  $a_{21} = a_{22} = a_{23} =: b_2$ , and therefore  $b_1 + b_2 = 2$ . Consequently (3.10) now becomes

$$\begin{aligned}
(S \otimes S \otimes S)_{(\alpha, \beta)}^A(\Psi) &= \frac{1}{2} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ b_1 (S_n^{(\beta)}(x))^2 S_{red} \left( x, \frac{1}{n^\alpha} \right) \right. \\
&\quad \left. + b_2 S_n^{(\beta)}(x) \left( S_{red} \left( x, \frac{1}{n^\alpha} \right) \right)^2 \right] \Psi(x) dx. \quad (3.11)
\end{aligned}$$

Finally, without going into detail, it is possible to prove that for the multiplication of four equal distributions we need to introduce three parameters  $c_1, c_2$ , and  $c_3$ , such that  $2c_1 + 3c_2 + 2c_3 = 7$ . The multiplication turns out to be

$$\begin{aligned}
& (S \otimes S \otimes S \otimes S)_{(\alpha, \beta)}^A(\Psi) \\
&= \frac{1}{7} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ 2c_1 (S_n^{(\beta)}(x))^3 S_{red} \left( x, \frac{1}{n^\alpha} \right) + 3c_2 (S_n^{(\beta)}(x))^2 \right. \\
&\quad \left. \times \left( S_{red} \left( x, \frac{1}{n^\alpha} \right) \right)^2 + 2c_3 S_n^{(\beta)}(x) \left( S_{red} \left( x, \frac{1}{n^\alpha} \right) \right)^3 \right] \Psi(x) dx. \quad (3.12)
\end{aligned}$$

The same procedure can be repeated even for a bigger number of distributions but we will omit this generalization here since the difficulty grows very fast with the number of distributions.

Just a comment before ending this section: even if this last method appears to be less natural than the first one, we will show in Section 4 that it works well in some examples, and its extra degrees of freedom may, in turn, be useful in future applications. The main difference between the two methods proposed in this section is that in the first one we increase

the number of indices  $\alpha$  and  $\beta$ , while in the second one we keep this number unchanged but we introduce new parameters which were not originally present in the definition we gave in [4]. As we have already observed, the preference must be given to that method whose results are closer to the experimental data, or to the common wisdom.

#### 4. EXAMPLES: DELTA FUNCTIONS

We devote this section to showing how the multiplications defined previously work explicitly. In particular, we will show that both methods proposed allow to define the product of an arbitrary number of delta functions in one dimension localized at the same point. The technique we are going to use is very much the same as the one used in [4] where two (derivatives of) delta functions have been shown to be multipliable. In particular we will need very often the well known Lebesgue dominated convergence theorem (LDCT); see [11] for example.

Before starting with the computation of the multiplications we remind the readers the expressions of the two regularizations of the delta function [4]. We have

$$\delta_n^{(\beta)}(x) \equiv n^\beta \Phi(n^\beta x), \quad \beta > 0 \quad (4.1)$$

and

$$\delta_{red}\left(x, \frac{1}{n^\alpha}\right) = \frac{1}{\pi n^\alpha} \frac{1}{\left(x^2 + \frac{1}{n^{2\alpha}}\right)}, \quad \alpha > 0. \quad (4.2)$$

We begin with considering the first method proposed, Eqs. (3.5)–(3.8), taking  $S = \delta$ . We fix first the form of the function generating the delta sequence. In this section we will always assume that  $\Phi(x)$  is the one given in (2.6), where  $F$  is a given normalization constant (of course  $m$ -depending) and  $m$  is an integer which must be taken even so to prevent  $\int_{-1}^1 \Phi(x) dx$  to be zero.

The result for  $(\delta \otimes \delta)_{(\alpha, \beta)}$  is already contained in Section 2; see (2.5). Changing the notation a little bit for future convenience, we have, for any  $\Psi \in \mathcal{D}(\mathbf{R})$ ,

$$(\delta \otimes \delta)_{(\alpha, \beta)}(\Psi) = \begin{cases} \frac{1}{\pi} A_{1,2} \delta(\Psi), & \alpha = 2\beta \\ 0, & \alpha > 2\beta, \end{cases} \quad (4.3)$$

where we have defined

$$A_{i,j} \equiv \int_{-1}^1 \frac{(\Phi(t))^i}{t^j} dt. \quad (4.4)$$

Of course, due to the presence of  $A_{1,2}$  in  $(\delta \otimes \delta)_{(\alpha, \beta)}(\Psi)$ , we need to take  $m \geq 2$ . Otherwise the integral defining  $A_{1,2}$  would be divergent.

This result coincides for both the methods proposed: this is obvious since the different multiplications introduced in the last section *both* generalize the multiplication discussed in [4] and refined in Section 2.

It is very easy to compute the product of three delta functions using our recipe; Eq. (3.6) now becomes

$$\begin{aligned} (\delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}(\Psi) &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta(x) \delta_n^{(\beta)}(x) \delta_{red}\left(x, \frac{1}{n^\alpha}\right) \Psi(x) dx \\ &= \frac{1}{\pi} \Phi(0) \lim_{n \rightarrow \infty} n^{\alpha+\beta} \Psi(0) = 0 \end{aligned}$$

since  $\Phi(0) = 0$  for any  $m > 0$ , for any choice of  $\alpha$  and  $\beta$  in  $\mathbf{R}_+$ .

Let us now move to the multiplication of four delta functions. The situation is no longer so easy. Using (3.7) we have

$$\begin{aligned} &(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)}(\Psi) \\ &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n^{(\beta_1)}(x) \delta_{red}\left(x, \frac{1}{n^{\alpha_1}}\right) \delta_n^{(\beta_2)}(x) \delta_{red}\left(x, \frac{1}{n^{\alpha_2}}\right) \Psi(x) dx. \end{aligned}$$

Here we are interested in showing that there exists a choice of  $m$ ,  $\alpha_i$ , and  $\beta_i$  for which the limit of the right hand side of this equation exists finite. We will show that such a result can be obtained already if we take  $\alpha_1 = \alpha_2 =: \alpha$  and  $\beta_1 = \beta_2 =: \beta$ , with some extra conditions on  $\alpha$  and  $\beta$ . We call  $(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}(\Psi) \equiv (\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)}(\Psi)$ . Introducing the variable  $t = xn^\beta$  in the integral, and using the fact that  $\Phi(t)$  has support in  $[-1, 1]$ , we obtain

$$(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}(\Psi) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n^{(\alpha, \beta)}(t) dt,$$

where

$$f_n^{(\alpha, \beta)}(t) \equiv \frac{1}{\pi^2 n^{2\alpha-5\beta}} \frac{(\Phi(t))^2 \Psi(t/n^\beta)}{(t^2 + 1/n^{2(\alpha-\beta)})^2}.$$

At this point we use the LDCT. In fact, for any  $\alpha$  and  $\beta$  with  $2\alpha \geq 5\beta$  we find that  $|f_n^{(\alpha, \beta)}(t)| \leq g(t)$ , where  $g(t) \equiv (LM/\pi^2 F)|t|^{2(m-2)}$ . Here we have used the same notation introduced in [4], and we have called  $M \equiv \sup_{t \in [-1, 1]} \exp\{1/t^2 - 1\}$  and  $L \equiv \sup_{t \in [-1, 1]} |\Psi(t)|$ . Of course,  $g(t)$  is integrable in  $[-1, 1]$  whenever  $m$  assumes values bigger than or equal to 2. Moreover, the function  $f_n^{(\alpha, \beta)}(t)$  converges pointwise, whenever  $2\alpha \geq 5\beta$ , to a function  $f^{(\alpha, \beta)}(t)$  which is equal to zero if  $2\alpha > 5\beta$  and to

$(\Phi(t))^2\Psi(0)/\pi^2t^4$  if  $2\alpha = 5\beta$ . In these conditions the LDCT can be applied and we get

$$(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}(\Psi) = \begin{cases} \frac{1}{\pi^2} A_{2,4} \delta(\Psi), & 2\alpha = 5\beta \\ 0, & 2\alpha > 5\beta, \end{cases} \quad (4.5)$$

where, of course,  $m \geq 2$ .

The  $(\otimes)_{(\alpha, \beta)}$  multiplication of five delta functions is again computed very simply. We have

$$\begin{aligned} & (\delta \otimes \delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha_1, \alpha_2, \beta_1, \beta_2)}(\Psi) \\ &= \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta(x) \delta_n^{(\beta_1)}(x) \delta_{red}\left(x, \frac{1}{n^{\alpha_1}}\right) \delta_n^{(\beta_2)}(x) \delta_{red}\left(x, \frac{1}{n^{\alpha_2}}\right) \Psi(x) dx = 0, \end{aligned}$$

using again the fact that  $\Phi(0) = 0$  whenever  $m > 0$ .

We are now ready to generalize these results: let  $l$  be a natural number. Therefore, for any  $\Psi \in \mathcal{D}(\mathbf{R})$ ,

$$\underbrace{(\delta \otimes \cdots \otimes \delta)}_{2l+1}_{(\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_l)}(\Psi) = 0 \quad (4.6)$$

for any choice of  $\alpha_i$  and  $\beta_i$  and for  $\Phi$  given by (2.6) with  $m > 0$ . On the other hand the multiplication of an even number,  $2l$ , of delta functions may give a nonzero (and finite!) result. It depends, in general, on  $\alpha_i$  and  $\beta_i$  with  $i = 1, 2, \dots, l$ ; see Eqs. (3.5) and (3.7). As we have already discussed for  $l = 2$ , it is actually enough to put  $\alpha_1 = \alpha_2 = \cdots = \alpha_l =: \alpha$  and  $\beta_1 = \beta_2 = \cdots = \beta_l =: \beta$ . We obtain

$$\begin{aligned} \underbrace{(\delta \otimes \cdots \otimes \delta)}_{2l}_{(\alpha, \beta)}(\Psi) &= \underbrace{(\delta \otimes \cdots \otimes \delta)}_{2l}_{(\alpha, \dots, \alpha, \beta, \dots, \beta)}(\Psi) \\ &= \begin{cases} \frac{1}{\pi^l} A_{l, 2l} \delta(\Psi), & l\alpha = (3l-1)\beta \\ 0, & l\alpha > (3l-1)\beta. \end{cases} \end{aligned} \quad (4.7)$$

Obviously,  $A_{l, 2l} < \infty$  only if  $m \geq 2$ .

We now move to the second definition of the multiplication we introduced in the last section. We show that this method also gives nontrivial results.

We know already that the multiplication of two delta functions is certainly well defined, since it coincides with the multiplication obtained following

the first procedure. In other words, we have

$$(\delta \otimes \delta)_{(\alpha, \beta)}^A(\Psi) = (\delta \otimes \delta)_{(\alpha, \beta)}(\Psi) = \begin{cases} \frac{1}{\pi} A_{l, 2} \delta(\Psi), & \alpha = 2\beta \\ 0, & \alpha > 2\beta, \end{cases} \quad (4.8)$$

and  $m$  must be bigger or equal to 2.

When we consider three delta functions we obtain, from (3.11),

$$\begin{aligned} (\delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}^A(\Psi) &= \frac{1}{2} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ b_1(\delta_n^{(\beta)}(x))^2 \delta_{red}\left(x, \frac{1}{n^\alpha}\right) \right. \\ &\quad \left. + b_2 \delta_n^{(\beta)}(x) \left( \delta_{red}\left(x, \frac{1}{n^\alpha}\right) \right)^2 \right] \Psi(x) dx. \end{aligned} \quad (4.9)$$

We will not give here all the details of this computation, which are very close to those discussed above. The steps are, more or less, the same: we change the variable in the integrals putting  $t = xn^\beta$ , we restrict the integration range due to the compact support of  $\Phi(t)$ , and then we use the LDCT which can be applied under certain conditions on  $\alpha, \beta$ , and  $m$ . For example, the first contribution in (4.9) converges to a finite quantity whenever  $\alpha \geq 3\beta$  and for  $m \geq 1$ . On the contrary, the second contribution is surely convergent for  $\alpha \geq 2\beta$  and for  $m \geq 4$ . Collecting these results we obtain that, for all  $m \geq 4$ , then

$$(\delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}^A(\Psi) = \begin{cases} \frac{b_1}{\pi} A_{2, 2} \delta(\Psi), & \alpha = 3\beta \\ 0, & \alpha > 3\beta, \end{cases} \quad (4.10)$$

which is, in general, different from the analogous result,  $(\delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}(\Psi) = 0$ , obtained using the first method.

To multiply four delta functions we refer to Eq. (3.12). For any  $\Psi \in \mathcal{D}(\mathbf{R})$  we have

$$\begin{aligned} &(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}^A(\Psi) \\ &= \frac{1}{7} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left[ 2c_1(\delta_n^{(\beta)}(x))^3 \delta_{red}\left(x, \frac{1}{n^\alpha}\right) + 3c_2(\delta_n^{(\beta)}(x))^2 \left( \delta_{red}\left(x, \frac{1}{n^\alpha}\right) \right)^2 \right. \\ &\quad \left. + 2c_3 \delta_n^{(\beta)}(x) \left( \delta_{red}\left(x, \frac{1}{n^\alpha}\right) \right)^3 \right] \Psi(x) dx. \end{aligned} \quad (4.11)$$

Now we need to estimate, using the usual techniques, three different contributions: the first is convergent whenever  $\alpha \geq 4\beta$  and for any natural  $m$ . The second one converges whenever  $2\alpha \geq 5\beta$  and  $m \geq 2$ . The last term, finally, converges if  $\alpha \geq 2\beta$  and  $m \geq 6$ . We conclude that, for any  $m \geq 6$ ,

$$(\delta \otimes \delta \otimes \delta \otimes \delta)_{(\alpha, \beta)}^A(\Psi) = \begin{cases} \frac{2c_1}{7\pi} A_{3, 2} \delta(\Psi), & \alpha = 4\beta \\ 0, & \alpha > 4\beta. \end{cases} \quad (4.12)$$

It may be worthwhile to notice that the condition on  $m$  does not follow from the requirement of  $A_{3,2}$ , to be finite. In fact  $A_{3,2} < \infty$  for any natural  $m$ . It follows from the analogous requirement for  $A_{1,6}$ , which appears in the computation of the last contribution in (4.11), the one proportional to  $c_3$ .

Of course an extra degree of freedom is present now: the coefficients  $b_1$  in (4.10) and  $c_1$  in (4.12) must satisfy only the very weak constraints:  $b_1 + b_2 = 2$  and  $2c_1 + 3c_2 + 2c_3 = 7$ . But, since  $b_2$ ,  $c_2$  and  $c_3$  do not appear at all, any choice of  $b_1$  and  $c_1$  is allowed.

We now generalize the above results. In general we get

$$\underbrace{(\delta \otimes \cdots \otimes \delta)}_l{}_{(\alpha, \beta)}^A(\Psi) = \begin{cases} \frac{d}{\pi} A_{l-1,2} \delta(\Psi), & \alpha = l\beta \\ 0, & \alpha > l\beta, \end{cases} \quad (4.13)$$

where  $d$  is a positive constant and  $m \geq 2(l-1)$ . Again, this constraint on  $m$  follows from a term which under these hypotheses on  $m$ ,  $\alpha$ , and  $\beta$  does not contribute to the final result, that is, the one proportional to  $\int_{-\infty}^{\infty} \delta_n^{(\beta)}(x) (\delta_{red}(x, 1/n^\alpha))^{l-1} \psi(x) dx$ .

*Remarks.* (a) It is interesting to observe that, for any odd integer  $n$ , the  $(\otimes)_{(\alpha, \beta)}^A$  multiplication of  $n$  delta functions may be different from zero, while the analogous computation made using  $(\otimes)_{(\alpha, \beta)}$  gives zero.

(b) It is straightforward to generalize all the results obtained in this section even to the multiplication of the derivatives of the delta function. The technique is, more or less, the same. We refer to [4] for the details on the regularization procedures of the distributions  $\delta^{(p)}(x)$ .

As in [4] we can apply these results to one dimensional physical models which describe media with impurities localized in certain fixed points, or to the discussion of the classical limit of a certain quantum mechanical situation. Let us consider, for instance, a three-particle system described by a factorizable wave function

$$\Phi^\epsilon(x_1, x_2, x_3, t) = \Phi_1^\epsilon(x_1, t) \Phi_2^\epsilon(x_2, t) \Phi_3^\epsilon(x_3, t),$$

where

$$|\Phi_1^\epsilon(x, 0)|^2 = |\Phi_2^\epsilon(x, 0)|^2 = |\Phi_3^\epsilon(x, 0)|^2 \equiv \frac{\exp\{-(x/\epsilon)^2\}}{\epsilon\sqrt{\pi}}.$$

We know that  $P_\epsilon(x_1, x_2, x_3) \equiv |\Phi_1^\epsilon(x_1, 0)|^2 |\Phi_2^\epsilon(x_2, 0)|^2 |\Phi_3^\epsilon(x_3, 0)|^2 dx_1 dx_2 dx_3$  is the probability of finding at  $t = 0$  particle  $i$  between  $x_i$  and  $x_i + dx_i$ ,  $i = 1, 2, 3$  [12]. In the limit  $\epsilon \rightarrow 0$  we get  $|\Phi_i^\epsilon(x, 0)|^2 \rightarrow \delta(x)$  (for instance in  $\mathcal{D}'$ ), so that  $P_\epsilon(x_1, x_2, x_3) \rightarrow \delta(x_1)\delta(x_2)\delta(x_3) dx_1 dx_2 dx_3$ .

Because of this we say that  $\epsilon \rightarrow 0$  corresponds to the classical limit of the system: in fact each particle is sharply centered in a single point.

We may look, therefore, for the probability of finding the three particles at the same point  $x$ , in this classical limit. Of course simple physical considerations require this probability to be zero. Therefore, since this probability should be proportional to  $(\delta(x))^3$ , we conclude that the natural regularization is the one in (4.6) with  $l = 1$  and for any choice of  $\alpha$  and  $\beta$ , or the one in (4.10) with  $\alpha > 3\beta$  and  $m \geq 4$ .

## 5. ANOTHER EXAMPLE: KLEIN-GORDON MODEL IN 1 + 1 DIMENSIONS

The model of free bosons which we are going to discuss in this section is defined by the second order differential equation

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \varphi(x, t) = 0 \quad (5.1)$$

and by the equal time canonical commutation relations

$$\begin{aligned} [\varphi(x, t), \varphi(x', t)] &= 0 \\ [\dot{\varphi}(x, t), \dot{\varphi}(x', t)] &= 0, \\ [\varphi(x, t), \dot{\varphi}(x', t)] &= i\delta(x - x'). \end{aligned} \quad (5.2)$$

Following the notation and the main steps of [13], we expand the solution of the Klein-Gordon equation in plane waves,

$$\varphi(x, t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} [a(k)e^{ikx-i\omega_k t} + a^\dagger(k)e^{-ikx+i\omega_k t}], \quad (5.3)$$

where  $\omega_k = \sqrt{k^2 + m^2}$  and the operators  $a(k)$  and its hermitean conjugate  $a^\dagger(k)$  are the coefficients of the expansion. They satisfy these canonical commutation relations:

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0, \quad [a(k), a^\dagger(k')] = \delta(k - k'). \quad (5.4)$$

Let us call  $\Psi_0$  the ground state of the theory [13]. This is defined by requiring that  $a(k)\Psi_0 = 0 \ \forall k \in \mathbf{R}$ .

Interesting quantities to compute are the expectation values in  $\Psi_0$  of the field  $\varphi(x, t)$  and of the product of the field,  $\varphi(x, t)\varphi(x', t')$ . As in the four dimensional situation, even in this simpler model problems arise when we try to compute the matrix element of the product  $\varphi(x, t)\varphi(x, t)$ . In particular we observe that

$$(\Psi_0, \varphi(x, t)\Psi_0) = 0,$$



while a straightforward calculation shows that

$$\Delta_+(r_x - r_y) \equiv (\Psi_0, \varphi(r_x)\varphi(r_y)\Psi_0) = \int_{-\infty}^{\infty} \frac{dk}{4\pi\omega_k} e^{-i\hat{k} \cdot (r_x - r_y)}, \quad (5.5)$$

where  $r_x = (x_0, -x)$  and  $\hat{k} \cdot (r_x - r_y) = \omega_k(x_0 - y_0) - k(x - y)$ . It is therefore evident that, in the limit  $r_x \rightarrow r_y$ ,  $\Delta_+(r_x - r_y)$  diverges logarithmically. (Recall that in four dimensions the analogous divergence is quadratic.)

Now we are ready to discuss the application of the regularizations proposed to the Klein-Gordon field. In particular we will discuss first the regularization of  $\varphi(x, t)$  when  $t$  is considered an extra parameter. This choice is necessary, at this stage of knowledge, since the analytical regularization has been introduced only in  $\mathbf{R}$ , while the sequential completion method is formulated in  $\mathbf{R}^n$ . The generalization of the analytic regularization to  $n > 1$  is discussed in [7]. Even if it is easily seen that both regularization procedures work well as far as the smearing of the field is concerned, we will also conclude that the multiplication discussed in Section 2 does not allow us to control the divergence of  $\Delta_+(0)$ , even if the time is considered properly and not as a parameter. We hope to be able to reconsider positively this problem in a future paper.

We start considering the analytic regularization of the field  $\varphi$ . Using definition (2.2) and considering  $t$  as a parameter, we get for any  $\epsilon > 0$

$$\begin{aligned} \varphi^0(x + i\epsilon, t) &\equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(y, t) dy}{y - (x + i\epsilon)} = \int_{-\infty}^0 \frac{dk}{\sqrt{4\pi\omega_k}} a^\dagger(k) e^{-ikx + i\omega_k t} e^{k\epsilon} \\ &+ \int_0^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} a(k) e^{ikx - i\omega_k t} e^{-k\epsilon}, \end{aligned}$$

where some easy applications of the integration in the complex domain have been used. Analogously we get

$$\begin{aligned} \varphi^0(x - i\epsilon, t) &= - \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} a(k) e^{ikx - i\omega_k t} e^{k\epsilon} \\ &- \int_0^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} a^\dagger(k) e^{-ikx + i\omega_k t} e^{-k\epsilon}. \end{aligned}$$

Therefore the regularized function,  $\varphi_{reg}(x, \epsilon; t) \equiv \varphi^0(x + i\epsilon, t) - \varphi^0(x - i\epsilon, t)$ , can be written as

$$\varphi_{reg}(x, \epsilon; t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} [a^\dagger(k) e^{-ikx + i\omega_k t} + a(k) e^{ikx - i\omega_k t}] P_\epsilon(k), \quad (5.6)$$

where we have introduced the (even) function  $P_\epsilon(k) \equiv e^{-k\epsilon} \theta(k) + e^{k\epsilon} \theta(-k)$ . From this equation and from (5.3) it is easy to understand

heuristically why  $\varphi_{reg}(x, \epsilon; t)$  is called a “regularization” of  $\varphi$ : it appears evident, in fact, that when  $\epsilon \rightarrow 0$  then  $\varphi_{reg}$  converges in some sense to  $\varphi$ . This follows from the fact that, when  $\epsilon \rightarrow 0$ , hence  $P_\epsilon(k) \rightarrow \theta(k) + \theta(-k)$ . Therefore, in this limit, this function behaves like the unit function whenever considered “inside an integral.” More precisely, if  $f(k)$  is an integrable function, we have

$$\int_{-\infty}^{\infty} dk f(k) \lim_{\epsilon \rightarrow 0} P_\epsilon(k) = \int_{-\infty}^{\infty} dk f(k).$$

Let us now make this heuristical argument rigorous, showing that  $\varphi_{reg}(x, \epsilon; t)$  converges to  $\varphi(x, t)$  in the topology of  $\mathcal{S}'(\mathbf{R})$  whenever  $\epsilon$  is sent to 0. In particular, we are going to show that, in the limit  $\epsilon \rightarrow 0$ , the quantity

$$\delta_\epsilon(\varphi) \equiv \left( \Psi_1, \int_{-\infty}^{\infty} [\varphi_{red}(x, \epsilon; t) - \varphi(x, t)] \zeta(x) dx \Psi_2 \right) \quad (5.7)$$

goes to zero. Here  $\Psi_1, \Psi_2$  are vectors of a dense subset of the Hilbert space, and  $\zeta(x)$  is a function in  $\mathcal{S}(\mathbf{R})$ . Using Eqs. (5.6) and (5.3), expliciting the form of  $P_\epsilon(k)$ , and introducing the functions  $a_{12}(k) \equiv (\Psi_1, a(k)\Psi_2)$ ,  $a_{12}^\dagger(k) \equiv (\Psi_1, a^\dagger(k)\Psi_2)$  and the Fourier transform of  $\zeta(x)$ ,

$$\tilde{\zeta}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(x) e^{ikx} dx,$$

we can write  $\delta_\epsilon(\varphi)$  as the sum of four contributions all, more or less, of the same kind. The first contribution, for instance, is proportional to

$$\int_0^\infty \frac{dk}{\sqrt{2\omega_k}} (e^{-k\epsilon} - 1) a_{12}(k) e^{-i\omega_k t} \tilde{\zeta}(k).$$

Since  $\tilde{\zeta}(k)$  is a function of  $\mathcal{S}$  and  $a_{12}(k)$  is surely well behaved, we can use LDCT to conclude that the above integral converges to zero when  $\epsilon$  goes to 0. We arrive at similar conclusions also for the other three contributions in  $\delta_\epsilon(\varphi)$ . This implies that  $\varphi_{red}(x, \epsilon; t)$  converges to  $\varphi(x, t)$  in  $\mathcal{S}'$ .

We discuss now the way in which a delta family can be used in the regularization of the scalar field. As for the analytic method we consider the time as a parameter. Therefore we have

$$\varphi_n^{(\beta)}(x, t) \equiv \int_{-\infty}^{\infty} \delta_n^{(\beta)}(y) \varphi(x - y, t) dy = \int_{-\infty}^{\infty} \Phi(q) \varphi\left(x - \frac{q}{n^\beta}, t\right) dq, \quad (5.8)$$

where, as usual, we indicate with  $\Phi(x)$  the function generating the  $\delta$ -sequence. We now prove explicitly that if  $\Phi$  satisfies the following three

conditions, then  $\varphi_n^{(\beta)}(x, t) \rightarrow \varphi(x, t)$  in  $\mathcal{S}'$ :

- (i)  $\int_{-\infty}^{\infty} \Phi(x) dx = 1$ ;
- (ii)  $\Phi(x) = \Phi(-x)$ ;
- (iii)  $\Phi \in \mathcal{S}(\mathbf{R})$ .

Incidentally, we observe that such a function generates a delta family of Dirichelet type by means of the procedure discussed in Section 2. Condition (ii), which is not required in the original definition of the functions of this family, is only a useful technical requirement.

Since  $t$  is considered as an extra parameter, we need to prove explicitly the convergence of  $\varphi_n^{(\beta)}(x, t)$  to  $\varphi(x, t)$ . For this reason, similarly to what we have done in (5.7), we compute the limit

$$\lim_{n \rightarrow \infty} \tilde{\delta}_n(\varphi) \equiv \lim_{n \rightarrow \infty} \left( \Psi_1, \int_{-\infty}^{\infty} [\varphi_n^{(\beta)}(x, t) - \varphi(x, t)] \zeta(x) dx \Psi_2 \right),$$

where  $\Psi_1, \Psi_2$ , and  $\zeta(x)$  are the same as in  $\delta_\epsilon(\varphi)$ . Using the parity of the function  $\Phi$  and introducing again the functions  $a_{12}(k)$  and of  $a_{12}^\dagger(k)$ , we can write  $\tilde{\delta}_n(\varphi)$  as the sum of two contributions with the same structure. In particular the first term of  $\tilde{\delta}_n(\varphi)$ ,  $\tilde{\delta}_{n,1}(\varphi)$  is

$$\tilde{\delta}_{n,1}(\varphi) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega_k}} a_{12}(k) e^{-i\omega_k t} \tilde{\zeta}(-k) \left( 2\pi \tilde{\Phi}\left(\frac{k}{n^\beta}\right) - 1 \right).$$

Again, we make use of the LDCT. The procedure is now a bit tricky. First of all, since  $\tilde{\zeta}$  belongs to  $\mathcal{S}$ , as well as  $\tilde{\Phi}$ , it is clear that,  $\forall n \in \mathbf{N}$ , the function

$$f_n(k) = \frac{1}{\sqrt{4\pi\omega_k}} a_{12}(k) e^{-i\omega_k t} \tilde{\zeta}(-k) \left( 2\pi \tilde{\Phi}\left(\frac{k}{n^\beta}\right) - 1 \right)$$

belongs to  $\mathcal{L}^1(\mathbf{R})$ . In fact we can write  $|f_n(k)| \leq g(k)$ , with

$$g(k) \equiv \frac{1}{\sqrt{4\pi\omega_k}} |a_{12}(k) \tilde{\zeta}(-k)| (2\pi M + 1).$$

Here  $M$  is the supremum of the function  $\tilde{\Phi}$ . Obviously, since  $g(k) \in \mathcal{L}^1(\mathbf{R})$ , then also  $f_n(k) \in \mathcal{L}^1(\mathbf{R})$ .

This implies that, for any  $\epsilon > 0$ , it is possible to choose a positive quantity  $R_\epsilon$ , independent on  $n$ , such that  $\int_{|k| > R_\epsilon} |f_n(k)| dk < \epsilon$ . Due to the hypothesis (i) of normalization of the function  $\tilde{\Phi}$ , which can also be written in terms of its Fourier transform as  $2\pi \tilde{\Phi}(0) = 1$ , we deduce that, as far as  $|k| \leq R_\epsilon$ ,  $f_n(k)$  surely converges almost everywhere to the function zero. This means that, using LDCT,

$$\lim_{n \rightarrow \infty} \int_{|k| \leq R_\epsilon} |f_n(k)| dk = \int_{|k| \leq R_\epsilon} \lim_{n \rightarrow \infty} |f_n(k)| dk = 0,$$

which also implies that, given  $\epsilon$ , there exists  $n_\epsilon \in \mathbb{N}$  such that, for all  $n > n_\epsilon$ ,  $|\int_{|k| \leq R_\epsilon} |f_n(k)| dk| < \epsilon$ . We can conclude that for all  $\epsilon > 0$ , there exists a natural  $n_\epsilon$  such that, for all  $n$  bigger than  $n_\epsilon$ ,

$$\left| \int_{-\infty}^{\infty} f_n(k) dx \right| < 2\epsilon.$$

An analogous estimate can be performed also for the second contribution of  $\tilde{\delta}_n(\varphi)$ ,  $\tilde{\delta}_{n,2}(\varphi)$ . We conclude that  $\varphi_n^{(\beta)}$  converges to  $\varphi$  in  $\mathcal{S}'$ .

Defining the set of functions

$$\mathcal{X} \equiv \left\{ \Psi(x) \in \mathcal{S}(\mathbb{R}) : \int_{-\infty}^{\infty} \Psi(x) dx = 1, \Psi(x) = \Psi(-x) \right\}, \quad (5.9)$$

we can summarize the above results in:

**PROPOSITION 2.** *For all  $\Phi \in \mathcal{X}$  the function  $\varphi_n^{(\beta)} = \int_{-\infty}^{\infty} \Phi(q) \varphi(x - q/n^\beta, t) dq$  converges to  $\varphi(x, t)$  in the topology of  $\mathcal{S}'$ .*

More results on this convergence will be discussed in the Appendix.

Once we have shown how the regularizations work for the quantum free field, we may think to use  $\varphi_n^{(\beta)}(x, t)$  and  $\varphi_{red}(x, \epsilon; t)$  to eliminate (some) divergences appearing in the quantum model. For instance we may think that the regularization of  $\Delta_+(r_x - r_y)$  can be made finite for  $r_x = r_y$ . Unfortunately this is not so. In fact, let us define, as it is natural,

$$[\Delta_+(r_x - r_y)]_{(\alpha, \beta)} \equiv (\Psi_0, (\varphi(r_x) \otimes \varphi(r_y))_{(\alpha, \beta)} \Psi_0), \quad (5.10)$$

and let us focus our attention in particular to  $[\Delta_+(0)]_{(\alpha, \beta)}$ .

We start computing (see (3.5))

$$I_n(\varphi) \equiv \left( \Psi_0, \varphi_n^{(\beta)}(x, t) \varphi_{reg} \left( x, \frac{1}{n^\alpha}; t \right) \Psi_0 \right),$$

and then we discuss the limit of  $I_n(\varphi)$  for  $n$  diverging. Considering only the nonvanishing contributions ( $a(k)\Psi_0 = \Psi_0 a^\dagger(k) = 0$ ), and using the commutation relations of the bosonic operators  $a(k)$  and  $a^\dagger(k)$ , we get

$$I_n(\varphi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dw \Phi(w) \int_{-\infty}^{\infty} \frac{dk}{\omega_k} e^{-ikw/n^\beta} e^{-\theta(k)k/n^\alpha}, \quad (5.11)$$

where  $\theta(k)$  is a function which is equal to 1 for  $k \geq 0$  and to  $-1$  otherwise.

Using the fact that, since  $\Phi(x)$  is taken in  $\mathcal{X}$  then it is an even function, as well as its Fourier transform, we have

$$I_n(\varphi) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{\omega_k} \tilde{\Phi} \left( \frac{k}{n^\beta} \right) e^{-\theta(k)k/n^\alpha} = \int_0^{\infty} \frac{dk}{\omega_k} \tilde{\Phi} \left( \frac{k}{n^\beta} \right) e^{-k/n^\alpha}. \quad (5.12)$$

Let us introduce now this new set of functions:

$$\tilde{\mathcal{X}}_0 \equiv \left\{ \tilde{\Psi}(k) \in \mathcal{D}(\mathbf{R}) : 2\pi\tilde{\Psi}(0) = 1, \tilde{\Psi}(k) = \tilde{\Psi}(-k) \right\}. \quad (5.13)$$

It is obvious that the inverse Fourier transform,  $FT^{-1}$ , of any function in  $\tilde{\mathcal{X}}_0$  belongs to  $\mathcal{X}$ , since  $\mathcal{D} \subset \mathcal{S}$ . It may be useful to take  $\Phi$  such that  $\tilde{\Phi} \in \tilde{\mathcal{X}}_0$ , since in this way  $I_n(\varphi)$  can be computed easily using numerical techniques. With the change of variable  $q = k/n^\beta$ , calling again  $M$  the supremum of the function  $\tilde{\Phi}(k)$ , and assuming that the support of  $\tilde{\Phi}(k)$  is the interval  $[-1, 1]$ , we deduce that

$$I_n(\varphi) = \int_0^1 \frac{dq}{\sqrt{q^2 + \frac{m^2}{n^{2\beta}}}} \tilde{\Phi}(q) e^{-qn^{\beta-\alpha}} \leq M \int_0^1 \frac{dq}{\sqrt{q^2 + \frac{m^2}{n^{2\beta}}}} e^{-qn^{\beta-\alpha}}. \quad (5.14)$$

In order to get information on the asymptotic behavior of  $I_n(\varphi)$  we begin with an easy estimate which shows that the integral in the r.h.s. cannot be convergent for  $n \rightarrow \infty$  whenever  $\alpha \geq \beta$ . This follows from the following analytic estimate: since for  $q \in [0, 1]$   $e^{-qn^{\beta-\alpha}} \geq e^{-n^{\beta-\alpha}}$ , it follows that

$$\begin{aligned} \int_0^1 \frac{dq}{\sqrt{q^2 + \frac{m^2}{n^{2\beta}}}} e^{-qn^{\beta-\alpha}} &\geq e^{-n^{\beta-\alpha}} \int_0^1 \frac{dq}{\sqrt{q^2 + \frac{m^2}{n^{2\beta}}}} \\ &= e^{-n^{\beta-\alpha}} \log \left( \frac{n^\beta + \sqrt{n^{2\beta} + m^2}}{m} \right). \end{aligned}$$

Of course, whenever  $\alpha \geq \beta$  the right hand side diverges. This does not really imply that  $I_n(\varphi)$  also diverges, as is clear. Nevertheless it is a very strong indication which, moreover, is also supplemented by the following remark: when  $n \rightarrow \infty$ , the first integral in (5.14) behaves, when  $\alpha \geq \beta$ , like  $\int_0^1 \frac{dq}{q} \tilde{\Phi}(q)$ , which can be finite only if  $\tilde{\Phi}(q)$  goes to zero when  $q \rightarrow 0$ . This is not what we have since  $\Phi$  belongs to  $\mathcal{X}$ , so that its value in  $k = 0$  is  $\frac{1}{2\pi}$ . These results suggest that for  $I_n(\varphi)$  to be converging,  $\beta$  must be chosen bigger than  $\alpha$ . But also in this case it is not easy to find an analytic estimate for the integral in (5.14) proving that  $\lim_{n \rightarrow \infty} I_n(\varphi) < \infty$ . For this reason we have used numerical procedures to compute this integral, for different choices of  $\alpha$  and  $\beta$ . Unluckily these numerical results seem to show again that  $\lim_{n \rightarrow \infty} I_n(\varphi) = \infty$ , even if the divergence is very slow.

Before ending this section, we briefly comment on the complete regularization of the field, that is, the one in which we consider properly  $t$  as the time coordinate of the field. First of all we notice that, in a certain sense, only in this case we are allowed to speak of a *canonical regularization* of the field since the general theory says that the two dimensional convolution  $\varphi * \delta_n^{(\beta)}$  is a  $C^\infty$  function and that the two dimensional analytic regularization is an analytic function. The computation of  $I_n(\varphi)$  does not present

many differences with respect to the situation discussed above and, by the way, the conclusion is still the same: we get, for any choice of  $\alpha$  and  $\beta$ ,  $\lim_{n \rightarrow \infty} I_n(\varphi) = \infty$ . For this reason we believe it is not worthwhile to give here the details of this procedure, which are much heavier than those discussed above and, again, do not lead to a positive conclusion.

## 6. CONCLUSIONS

In this paper we have discussed different generalizations of the multiplication of distributions first introduced in [4]. In particular, we have proposed possible modifications of the sequential completion method which may be of some utility depending on the distribution to be regularized.

Furthermore, we have introduced two different definitions of multiplications of  $N > 2$  distributions, both of which generalize the definition given in Section 2 for  $N = 2$ . Of course, many other generalizations are also possible. We have shown how both these definitions can be used to define the multiplications of an arbitrary number of delta functions localized all at the same point. A quantum mechanical physical example has been also sketched.

Finally we have discussed a naive possibility of using our strategy in QFT. We have shown that it is possible to regularize the quantum field in many ways, but unfortunately the definition proposed in Section 2 does not allow us to cancel out the divergence appearing already for a free theory. Our future projects are therefore to look for some refinement of the procedure which allows us to overcome this last problem. If this new technique can be found, we can also try to extend the theory to four dimensional models and to discuss the divergences coming from the Feynman graphs. The final aim is to consider a nonabelian gauge theory like QCD [14].

In this analysis we expect that a crucial role will be played by the function  $\Phi$  and by the parameters  $\alpha$  and  $\beta$  which fix the multiplication. They should have the same role as the free parameters in renormalization theory, whose values are fixed by the experimental data.

## APPENDIX: A CONVERGENCE REMARK

In this appendix we prove in a different (and easier) way that, whenever  $\Phi(x) \in \mathcal{L}$ , then  $\varphi_n^{(\beta)}(x, t) \rightarrow \varphi(x, t)$  in  $\mathcal{S}'$ .

Let  $\zeta(x) \in \mathcal{S}(\mathbf{R})$ . After some easy computation we deduce that

$$\begin{aligned} A_n^\beta &\equiv \int_{-\infty}^{\infty} \varphi_n^{(\beta)}(x, t) \zeta(x) dx - \int_{-\infty}^{\infty} \varphi(x, t) \zeta(x) dx \\ &= \int_{-\infty}^{\infty} \left( \delta_n^{(\beta)}(y) - \delta(y) \right) \int_{-\infty}^{\infty} \varphi(x - y, t) \zeta(x) dx dy. \end{aligned}$$

Of course the integral  $\eta(y) \equiv \int_{-\infty}^{\infty} \varphi(x-y, t) \zeta(x) dx$  is continuous in  $y$ . Using the results in [8], we can conclude that  $A_n^\beta \rightarrow 0$  for  $n \rightarrow \infty$ . In fact, in particular, if  $\Phi(x)$  is taken positive, then it generates a delta family of positive type, so that for any function  $f(x)$  continuous in the origin we have

$$\int_{-\infty}^{\infty} \delta_n^{(\beta)}(x) f(x) dx \rightarrow f(0).$$

If  $\Phi$  is not positive the same conclusion still holds since  $\eta(y)$  is also differentiable in  $y = 0$ ; see [8].

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